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DETERMINATION OF INVARIANT MEASURES: AN APPROACH BASED ON HOMOTOPY PERTURBATIONS

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This paper describes the application of the homotopy perturbations method (HPM) in the computation of invariant measures (IMs) of the non-linear dynamical systems which are characterized by the complex, chaotic behavior. The convergence of the HPM is formally investigated and confirmed, and its efficiency is illustrated in several examples of widely used chaotic maps.

Keywords: Invariant measures, Frobenius-Perron equation, homotopy perturbations, chaotic mappings, probability distribution.

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1. Introduction

One of the main problems in the study of nonlinear dynamical models consists in the determination of their potential stochastic characteristics. Even in the case of the nonlinear models of a deterministic type, particular attention is dedicated to analysis of the statistical properties of their realizations (orbits). As the basic indicators of their asymptotic behavior, for different choices of starting points, the invariant (probabilistic) measures (IMs) are commonly used. The importance of IMs is particularly apparent in the precise analyses of the so-called chaotic dynamical models. However, the determination of IMs is usually based on the famous Frobenius-Perron equation, for which there exists no general procedure. In this aim, here is described one of the possible ways of solving these kinds of problems, based on homotopy perturbations method (HPM).

The HPM, proposed by He [1]-[4], is a general approximate-analytical approach, often used to obtain the solutions of nonlinear equations of various types. In the recent years, this method has been subject of extensive studies [5]-[7], and has been applied in solving of the different kinds of problems such as various types of nonlinear differential and partial-differential equations [8]-[10], Fredholm and Volterra integral equations [11]-[13], as well as the different kinds of physical problems [14]-[16]. On the other side, the HPM has not found significantly applications in the theory of chaos, as well as in the stochastic theory in general. We point out that one similar, Homotopy Analysis Method (HAM) has been applied in approximation of infinity convolutions of mixed stochastic distributions in [17].

This paper is organized as follows. A brief theoretical background, i.e., definitions and the basic facts about chaotic maps and their invariant measures are given in following Section 2. The main results, compared to the application of HPM in solving the Frobenius-Perron equation, are presented in Section 3. In Section 4, we consider practical applications of the HPM procedure in finding IMs of some typical, widely used chaotic maps. Finally, Section 5 contains some concluding remarks.

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2. **Preliminaries & definition of the problem**

In general form, the non-linear dynamical model can be defined by the operator $T : A \to A$, where $A \subseteq \mathbb{R}^m$ and m is a dimension of operator T. More precisely, the operator T generates the recurrence sequence:

$$\mathbf{x}_{n+1}^{(m)} = T(\mathbf{x}_n^{(m)}), \quad n = 0, 1, 2, \dots \tag{1}$$

where $\mathbf{x}_n^{(m)} = (x_n, x_{n-1}, \dots, x_{n-m+1}) \in A$, and the properties of this sequence depend on the mathematical structure of operator T. If for example the sequence $(\mathbf{x}_n^{(m)})$ converges, its limit $\mathbf{x}^* := \lim_{n\to\infty} \mathbf{x}_n^{(m)}$ is uniquely determined by the equation $\mathbf{x}^* = T(\mathbf{x}^*)$. The value $\mathbf{x}^*$ is *the fixed point* of the operator T, and sufficient conditions for its existence give a well-known Banach fixed point theorem. On the other hand, the sequence $(\mathbf{x}_n^{(m)})$ can have two or more accumulation points, while in a limit case the number of such points becomes infinite. Thus, these sequences are called *chaotic*, and they can be formally defined in the following way (for more details, see for instance [18, 19]):

Definition 2.1. The operator $T : A \to A$, $A \subseteq \mathbb{R}^m$ generates a *dynamical chaos* if:

(i) T is *sensitive to the initial conditions*, i.e. there exist $\delta > 0$ such that for all $\mathbf{x} \in A$ and any neighborhood $U \ni \mathbf{x}$, there exist $\mathbf{y} \in U$ and integer $n \geq 0$ such that inequality $\|T^n(\mathbf{x}) - T^n(\mathbf{y})\| > \delta$ holds;

(ii) T is *topologically transitive*, i.e. if for all open sets $U, V \subseteq A$ exists integer $n > 0$ such that $T^n(U) \cap V \neq \emptyset$.

In free interpretation, the sensitivity of the operator $T(\mathbf{x})$ implies existence of trajectories with "closely" starting points, which become significantly different during subsequent realizations. Left panel in Fig. 1 shows typical behavior of the one dimensional operator $T(x) = 1 - |1 - 2x|$, defined in the interval $A = (0, 1)$. On the other hand, a property of transitivity means that the trajectories of chaotic dynamical systems will take a values in each open "part" of the set A in which they are defined. Such situation is shown in right panel in the Fig 1, where $A = (-1, 1)$ and the corresponding operator is $T(x) = 1 - 2\sqrt{|x|}$.

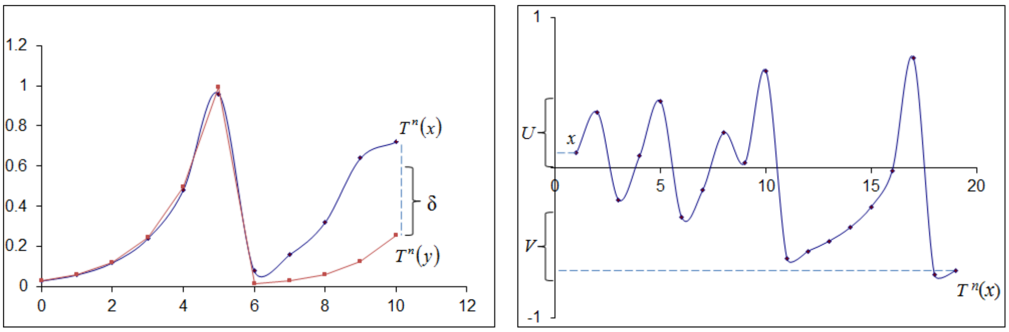

FIGURE 1. Graphical illustration of the properties of sensitivity (panel left) and transitivity (panel right) of the operator $T(\mathbf{x})$.

Now, we introduce another term which is close to sensitivity conditions, and represents important indicator of chaotic behavior of dynamical systems.

Definition 2.2. *Let* $\{\mathbf{x}_n^{(m)}\}_{n=0}^{\infty}$ *and* $\{\mathbf{y}_n^{(m)}\}_{n=0}^{\infty}$ *be a sequences in* $A \subseteq \mathbb{R}^m$*, defined by recurrence relation* (1)*, i.e.* $\mathbf{x}_n^{(m)} = T(\mathbf{x}_{n-1}^{(m)})$ *and* $\mathbf{y}_n = T(\mathbf{y}_{n-1}^{(m)})$*,* $n \geq 1$*. Then, the maximum Lyapunov exponent of the operator* $T(\mathbf{x})$ *is*

$$\lambda := \lim_{n\to\infty} \lim_{d_0\to 0} \frac{1}{n} \ln \frac{d_n}{d_0}, \tag{2}$$

where $d_n := \left\| \mathbf{x}_n^{(m)} - \mathbf{y}_n^{(m)} \right\|$, $n = 0, 1, 2 \ldots$

It was shown that Eq.(2), under certain conditions, uniquely determines Lyapunov exponent λ independent of the choice of initial values $\mathbf{x}_0^{(m)}, \mathbf{y}_0^{(m)}$. Also, if the operator $T(\mathbf{x})$ is a Frećhet-differentiable, then

$$\mathbf{x}_n^{(m)} - \mathbf{y}_n^{(m)} = T^n(\mathbf{x}_0^{(m)}) - T^n(\mathbf{y}_0^{(m)}) \approx DT^n(\mathbf{x}_0^{(m)})(\mathbf{x}_0^{(m)} - \mathbf{y}_0^{(m)}), \quad d_n \to 0,$$

where $DT^n(\mathbf{x}_0^{(m)})$ is the Jacobian of $T^n(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}_0^{(m)}$. Thus, the limit in Eq.(2) can be rewritten as

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \ln \left\| DT^n(\mathbf{x}_0^{(m)}) \right\|. \tag{3}$$

As is already mentioned, for chaotic dynamical systems, characterized by sensitivity to the initial conditions, there exist very rapid, exponential changes of their trajectories, even for "closely" initial values $\mathbf{x}_0^{(m)} \approx \mathbf{y}_0^{(m)}$. Thus, positive values $\lambda > 0$ indicates the chaotic structure of the sequence $\{\mathbf{x}_n^{(m)}\}$, i.e., of the operator $T(\mathbf{x})$.

In the following, we assume that $m = 1$, and we define some relevant concepts related to the definition of the problem of invariant measures (IMs) determination.

Definition 2.3. Let $(A, \mathcal{F}, \mu)$ be a probability space on $A \subseteq \mathbb{R}$, and let $T : A \to A$ be a measurable operator, i.e. $T^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{F}$. Then, T is a *measure-preserving* or, equivalently, μ is a T*-invariant measure*, if $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{F}$.

Now, suppose that the invariant measure μ is absolutely continuous with respect to the Lebesgue measure, i.e. that exist a density $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mu(B) = \int_B f(x)\,\mathrm{d}x, \quad \forall B \in \mathcal{F}.$$

Then, the properties of measure-preserving for $T(x)$ can be rewritten by the equality

$$\int_B T \circ f(x)\,\mathrm{d}x = \int_B f(x)\,\mathrm{d}x, \quad \forall B \in \mathcal{F}.$$

Definition 2.4. Let $(A, \mathcal{F}, \mu)$ be a probability space on bounded set $A \subset \mathbb{R}$, and $T : A \to A$ measure-preserving operator. In addition, assume that $T(x)$ is piecewise smooth, i.e. there exist a finite set of disjoint open intervals $I_1, \ldots, I_s$ such that $A \subseteq \overline{I}_1 \cup \cdots \cup \overline{I}_s$, where $\overline{I}_j$ is the closure form of I_j, and $T(x)$ is C^∞ map on each I_j, $j = 1, \ldots, s$. Then, $T(x)$ is *expanding map* if there are finite numbers $\beta > \alpha > 1$ such that $\beta \geq |T'(x)| \geq \alpha$, for each $x \in I_j$, $j = 1, \ldots, s$.

It can be shown (see, for instance [20]) that for piecewise smooth and expanding map $T(x)$, the density $f(x)$ of the absolutely continuous T-invariant measure μ obeys the well-known *Frobenius-Perron equation*:

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}. \tag{4}$$

Here, $T^{-1}(x) = \{g_1(x), \ldots, g_\ell(x)\}$ is a finite set of inverse branches of the operator $T(x)$. The existence of the solution of the Eq.(4), in the case of piecewise monotonic, continuous and convex maps on the unit interval $[0, 1]$ is proved in [21, 22]. On the other hand, the approximate solutions of Eq.(4) are mainly based on some finite approximation methods [23]-[25], or random perturbations based methods [26]-[28].

3. HPM Solution of the Frobenius-Perron Equation

In order to effective solve the Frobenius-Perron Eq.(4), here we propose a procedure based on the HPM. Firstly, we construct the following *homotopy equation*:

$$(1-p)\Big[F(x;p)-f_0(x)\Big]+h\,p\left[F(x;p)-\sum_{y\in T^{-1}(x)}\frac{F(y;p)}{|T'(y)|}\right]=0, \tag{5}$$

where $p\in(0,1)$ is *the embedding parameter*, and $h\neq 0$ is *the auxiliary parameter*. When $p=0$, the homotopy Eq.(5) has the so-called *initial solution* $F(x;0)=f_0(x)$, which can be chosen arbitrarily. On the other hand, when $p=1$, the solution of the homotopy Eq.(5) becomes equivalent to original Eq.(4). The basic assumption of HPM is that the solution of the homotopy Eq.(5) can be expressed as the power series in p:

$$F(x;p)=\sum_{k=0}^{\infty}p^k\,f_k(x). \tag{6}$$

In accordance to Eq.(5) and the condition of convergence the power series in Eq.(6), solution of the Frobenius-Perron Eq.(4) can be obtain as:

$$f(x):=\lim_{p\to 1^-}F(x;p)=\sum_{k=0}^{\infty}f_k(x). \tag{7}$$

Now, substituting the power series defined by Eq.(6) in Eq.(5), and after rearranging of some terms, the following equation holds:

$$\sum_{k=1}^{+\infty}p^k\,f_k(x)+h\,pf_0(x)=(1-h)\sum_{k=2}^{+\infty}p^kf_{k-1}(x)+h\sum_{k=1}^{+\infty}p^k\sum_{y\in T^{-1}(x)}\frac{f_{k-1}(y)}{|T'(y)|}. \tag{8}$$

Equating expressions in Eq.(8) with the identical powers p^k, $k=1,2,\dots$, we obtain the following equations:

$$f_1(x) = \sum_{y\in T^{-1}(x)}\frac{f_0(y)}{|T'(y)|}-h\,f_0(x), \tag{9}$$

$$f_k(x) = (1-h)\,f_{k-1}(x)+h\sum_{y\in T^{-1}(x)}\frac{f_{k-1}(y)}{|T'(y)|},\quad k\geq 2. \tag{10}$$

Using Eqs.(9)–(10), functions $\{f_k(x)\}$ can be obtained recursively for an arbitrary $k=1,2,\dots$. According to these, the so-called *HPM-approximations* of unknown function $f(x)$ will be:

$$\widehat{f}_k(x):=\sum_{j=0}^{k}f_j(x),\quad k=0,1,2,\dots \tag{11}$$

The convergence of this iterative procedure, under certain necessary conditions, will be proven in the following.

Theorem 3.1. Let $\{f_k(x)\}_{k=0}^{\infty}$ be the sequence of the non-negative functions defined on the bounded set $A\subset\mathbb{R}$ by recurrence relations in Eqs.(9)–(10). In addition, let assume that the following conditions are satisfied:

(i) The function $f_0(x)$ is bounded on A, i.e. there exist $M>0$ such that the inequality

$$\|f_0(x)\|:=\max_{x\in A}|f_0(x)|\leq M$$

holds.

(ii) $T(x)$ is a bounded, piecewise smooth and expanding operator on A, with finite set of ℓ inverse branches.

(*iii*) There exist $q > 0$ such that the inequalities $\left\|T'(x)^{-1}\right\| \leq q$ and $0 < h(1-q\,\ell) < 1$ hold.

Then, the sequence $\{\widehat{f}_k(x)\}_{k=0}^{\infty}$, defined by Eq.(11), uniformly converges on A to the function $f(x)$, i.e. to the solution of the Eq.(4).

Proof. According to assumptions of theorem, as well as Eqs.(9)-(10), for fixed but an arbitrary $x \in A$ we have:

$$\begin{aligned} |f_1(x))| &= \left| \sum_{y \in T^{-1}(x)} \frac{f_0(y)}{|T'(y)|} - h\, f_0(x) \right| \leq \sum_{y \in T^{-1}(x)} \frac{|f_0(y)|}{|T'(y)|} \leq q\,\ell M, \\ |f_2(x)| &\leq (1-h)\,|f_1(x)| + h \sum_{y \in T^{-1}(x)} \frac{|f_1(y)|}{|T'(y)|} \leq q\,\ell M B, \end{aligned}$$

where $B := 1 - h(1 - q\,\ell) \in (0,1)$. In general, using the induction method, it can be easily proved that inequalities:

$$|f_k(x)| \leq q\,\ell M B^{k-1} \tag{12}$$

hold for each $k = 1, 2, \ldots$ Now, let $r(x)$ be the radius of convergence of the power series in Eq.(6). Applying the Cauchy-Hadamard theorem and Eq.(12), it follows:

$$r(x) = \left[\limsup_{k\to\infty} |f_k(x)|^{1/k}\right]^{-1} \geq \lim_{k\to\infty} (q\,\ell M)^{-1/k} B^{-1+1/k} = B^{-1} > 1.$$

Thus, this power series converges at $p = 1$. On the other hand, according to Eqs.(9)-(11) we obtain:

$$\begin{aligned} \left|\widehat{f}_0(x)\right| &= |f_0(x)| \leq M, \\ \left|\widehat{f}_1(x)\right| &\leq \left|\widehat{f}_0(x)\right| + |f_1(x)| = M(1 + q\,\ell), \\ \left|\widehat{f}_2(x)\right| &\leq \left|\widehat{f}_1(x)\right| + |f_2(x)| = M(1 + q\,\ell + q\,\ell B), \end{aligned}$$

and, in general,

$$\left|\widehat{f}_k(x)\right| \leq M\left(1 + q\,\ell \sum_{j=0}^{k-1} B^j\right) = M\left(1 + q\,\ell \frac{1 - B^k}{1 - B}\right), \quad k = 1, 2, \ldots$$

In the limite case, when $k \to \infty$, it follows that:

$$\begin{aligned} \left|\sum_{j=0}^{\infty} f_j(x)\right| &= \lim_{k\to\infty} \left|\widehat{f}_k(x)\right| \leq M \lim_{k\to\infty} \left(1 + q\,\ell \frac{1 - B^k}{1 - B}\right) \\ &= M\left(1 + \frac{q\,\ell}{h(1 - q\,\ell)}\right) < +\infty. \end{aligned}$$

Therefore, the power series $\sum_{j=0}^{\infty} p^j f_j(x)$ is absolutely and uniformly convergent at $p = 1$. According to Abel's theorem, function $F(x,p)$, defined by Eq.(6), is continuous from the left at $p = 1$, i.e. the Eq.(7) holds. Thus, the series $\sum_{j=0}^{\infty} f_j(x)$ is a solution of the homotopy equation Eq.(5) when $p = 1$, i.e. it is solution of the Frobenius-Perron Eq.(4). □

Using the previous theorem, the errors of HPM-approximations can be estimate in the following way:

Corollary 3.1. *The error of k-th HPM approximation* $E_k(x) := \left| f(x) - \widehat{f}_k(x) \right|$ *for an arbitrary* $x \in A$ *satisfies inequality:*

$$E_k(x) \le \frac{q\,\ell M\,[1 - h(1 - q\,\ell)]^k}{h(1 - q\,\ell)}.$$

Proof. This statement follows immediately from the inequalities:

$$E_k(x) \quad = \quad \left| \sum_{j=0}^{\infty} f_j(x) - \sum_{j=0}^{k} f_j(x) \right| \le \sum_{j=k+1}^{\infty} |f_j(x)| \le q\,M \sum_{j=k}^{\infty} B^j = \frac{q\,MB^k}{1 - B}.$$

□

Remark 3.1. Thanks to the appropriate choice of the initial approximation $f_0(x)$, such that normalized condition $\int_a^b f_0(x)\,\mathrm{d}x = 1$ hold, the boundary condition (i) of the previous theorem can be easily fulfilled. On the other hand, condition (ii) ensures the existence of finite set $T^{-1}(x)$ of ℓ inverse branches, which are bounded on $A \subset \mathbb{R}$. Finally, notice that for chaotic maps, in accordance to Eq.(3), the condition $\min_{x \in A} |T'(x)| > 1$ ensures the positive value of Lyapunov exponent λ. Moreover, if $\|T'(x)\| > \ell$, the condition (iii) will be satisfied for an arbitrary $h \in (0, 1]$, and q such that

$$\left\| T'(x)^{-1} \right\| = \left\| (T^{-1}(x))' \right\| \le q < 1/\ell.$$

In the following, we give some application of the aforementioned HPM procedure, where these conditions will be examined.

4. Application of the HPM procedure

In this section, the practical application of the HPM in determining IMs is described on a several examples of widely used chaotic maps.

Example 4.1 (Tent map). Consider on the unit interval $(0, 1)$ the following map (Fig. 2, left panel):

$$T(x) = 1 - |1 - 2x| = \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2(1 - x), & 1/2 \le x \le 1. \end{cases}$$

Here, we have $|T'(x)| = \ell = 2$ and $T^{-1}(x) = \{x/2, 1 - x/2\}$, so the Frobenius-Perron equation is

$$f(x) = \frac{1}{2}\left[f\left(\frac{x}{2}\right) + f\left(1 - \frac{x}{2}\right) \right].$$

Although the condition (iii) of Theorem 3.1 is not fulfilled, taking $h = 1$, $f_0(x) \equiv 1$, and applying Eq.(9), it is obtained:

$$f_1(x) = \frac{1}{2}\left[f_0\left(\frac{y}{2}\right) + f_0\left(1 - \frac{y}{2}\right) \right] - f_0(x) = \frac{1}{2}(1 + 1) - 1 \equiv 0.$$

Then, Eq.(10) immediately gives $f_k(x) \equiv 0$, for all $k = 2, 3, \ldots$ Thus, the initial approximation $f_0(x) \equiv 1$ is the exact solution of Eq.(4), i.e. it is the invariant measure for this map.

Example 4.2 (Baker's map). Now, consider the closed unit interval $[0, 1]$ and the map (Fig. 2, right panel):

$$T(x; a) = \begin{cases} \dfrac{x}{a}, & 0 \le x < a, \\ \dfrac{a(x - a)}{(1 - a)}, & a \le x \le 1, \end{cases}$$

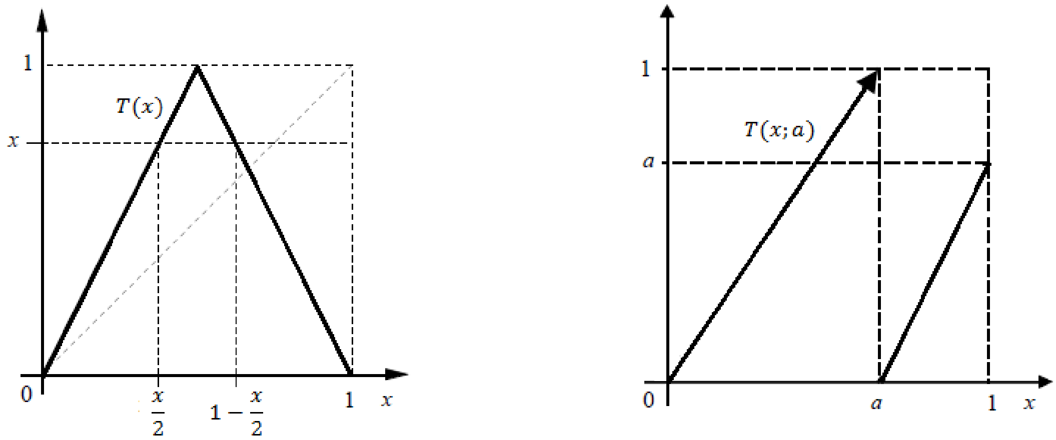

FIGURE 2. Graphics of the tent map (panel left) and the Baker's map (panel right).

where $a \in (0,1)$ is a predefined parameter. In this case, we have:

$$|T'(x)| = \begin{cases} \dfrac{1}{a}, & 0 \le x < a, \\ \dfrac{a}{1-a}, & a \le x \le 1, \end{cases}$$

and $T^{-1}(x) = \{g_1(x), g_2(x)\}$, where $g_1(x) = ax$, $0 \le x \le 1$, and

$$g_2(x) = \begin{cases} \dfrac{1-a}{a}x + a, & 0 \le x < a, \\ 0, & a \le x \le 1, \end{cases}$$

are the inverse branches of the operator $T(x;a)$. Thus, the Frobenius-Perron Eq.(4) takes a form:

$$f(x) = \begin{cases} a\, f(g_1(x)) + \dfrac{1-a}{a}\, f(g_2(x)), & 0 \le x < a, \\ a\, f(g_1(x)), & a \le x \le 1. \end{cases}$$

If we take, as in the previous example, $h = 1$ and $f_0(x) \equiv 1$, by applying Eqs.(9)-(10) we obtain:

$$\begin{aligned} f_1(x) &= \begin{cases} a\, f_0(g_1(x)) + \dfrac{1-a}{a}\, f_0(g_2(x)) - 1, & 0 \le x < a, \\ a\, f_0(g_1(x)) - 1, & a \le x \le 1, \end{cases} \\ &= \begin{cases} \dfrac{(a-1)^2}{a}, & 0 \le x < a, \\ a - 1, & a \le x \le 1, \end{cases} \\ f_2(x) &= \begin{cases} a\, f_1(g_1(x)) + \dfrac{1-a}{a}\, f_1(g_2(x)), & 0 \le x < a, \\ a\, f_1(g_1(x)), & a \le x \le 1, \end{cases} \\ &= \begin{cases} \dfrac{(a-1)^3}{a}, & 0 \le x < a, \\ (a-1)^2, & a \le x \le 1. \end{cases} \end{aligned}$$

In general, for each $k = 1, 2 \dots$ we have:

$$f_k(x) = \begin{cases} \dfrac{(a-1)^{k+1}}{a}, & 0 \le x < a, \\ (a-1)^k, & a \le x \le 1, \end{cases}$$

and according to these, it follows:

$$f(x) := \sum_{k=0}^{\infty} f_k(x) = \begin{cases} 1 + \dfrac{(a-1)^2}{a} \displaystyle\sum_{k=0}^{\infty} (a-1)^k, & 0 \le x < a, \\ 1 + (a-1) \displaystyle\sum_{k=0}^{\infty} (a-1)^k, & a \le x \le 1, \end{cases}$$

$$= \begin{cases} \dfrac{1}{a(2-a)}, & 0 \le x < a, \\ \dfrac{1}{2-a}, & a \le x \le 1. \end{cases}$$

It can be easily proved that the last expression is exact solution of the Frobenius-Perron equation. Thus, using the HPM is obtained the IM of the Baker's map. Notice that, as in the previous example, the condition (iii) of Theorem 3.1 cannot be fulfilled, but the HPM procedure converges for an arbitrary $a \in (0, 1)$.

Example 4.3 (**Logistic map**). At last, consider the map $T(x) = 4x(1-x)$, $x \in (0, 1)$, for which the set of inverse branches is $T^{-1}(x) = \{g_1(x), g_2(x)\}$, with $g_{1,2}(x) = (1 \pm \sqrt{x})/2$. Then, $|T'(g_{1,2}(x))| = 4\sqrt{1-x}$, and the Frobenius-Perron equation is

$$f(x) = \frac{f(g_1(x)) + f(g_2(x))}{4\sqrt{1-x}}.$$

If starting again with $h = 1$ and $f_0(x) \equiv 1$, the Eqs.(9)–(10) give:

$$f_1(x) = \frac{f_0(g_1(x)) + f_0(g_2(x))}{4\sqrt{1-x}} - f_0(x) = \frac{1}{2\sqrt{1-x}} - 1,$$

$$f_2(x) = \frac{f_1(g_1(x)) + f_1(g_2(x))}{4\sqrt{1-x}} = \frac{\sqrt{2 - 2\sqrt{1-x}} + \sqrt{2 + 2\sqrt{1-x}} - 2\sqrt{x}}{4\sqrt{x(1-x)}},$$

etc.

In general, by using the induction method, it can be prove that:

$$f_k(x) = \frac{Q_k(x)}{2^k \sqrt{x(1-x)}} - \widehat{f}_{k-1}(x), \quad k = 2, 3, \dots$$

where

$$Q_k(x) := \sum_{(j_1, j_2, \dots, j_k)} \sqrt{2 + (-1)^{j_1} \sqrt{2 + (-1)^{j_2} \sqrt{2 + \dots + (-1)^{j_{k-1}} 2\sqrt{1-x}}}},$$

$(j_1, j_2, \dots, j_{k-1})$ are the $(k-1)$-element variations of $\{0, 1\}$ with repetition, and $\widehat{f}_k(x)$ are the HPM-approximations defined by Eq.(11). Thus, when $k \to \infty$,

$$Q_k(x) \sim 2^{k-1} \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots + 2\sqrt{1-x}}}}}_{k \text{ times}}.$$

and applying Vieta formula, it follows $Q_k/2^k \sim 1/\pi$, $k \to \infty$. Finally,

$$f(x) = \lim_{k\to\infty} \widehat{f}_k(x) = \lim_{k\to\infty} \sum_{j=1}^{k} f_j(x) = \lim_{k\to\infty} \frac{Q_k(x)}{2^k\sqrt{x(1-x)}} = \frac{1}{\pi\sqrt{x(1-x)}},$$

where the limit function thus obtained is a exact solution of the Frobenius-Perron equation, i.e. represents the IM of the logistic map. Let us remark that this IM belongs to the well-known class of beta distributions, whose density, as well as the histogram of empirical distribution is shown in the left panel of Fig. 3. In the right panel of the same figure, convergence of the HPM-approximations can be seen. Notice that already for $k \geq 3$, the functions $\widehat{f}_k(x)$ give precise approximations of the IM $f(x)$.

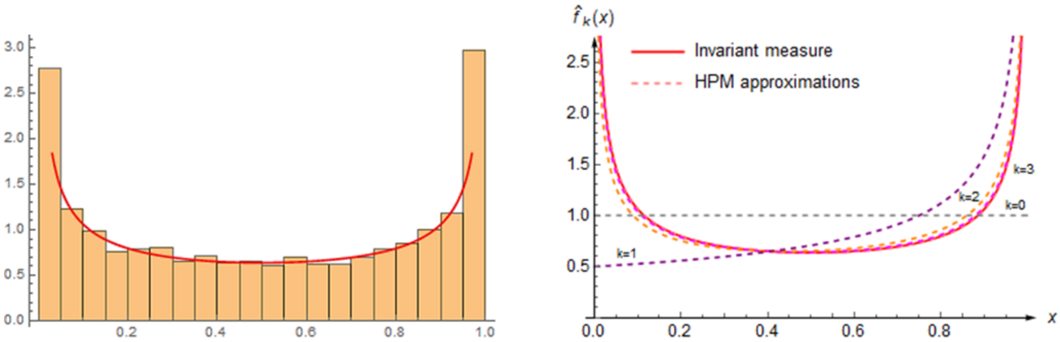

FIGURE 3. Histogram of 5.000 realizations of the logistic map (panel left) and the HPM-approximations of its invariant measure (panel right).

5. Conclusion

In this work the Homotopy Perturbation Method (HPM) has been used for solving Frobenius-Perron equation, i.e. for determination of the invariant measures in chaotic dynamical systems. Presented theoretical and practical results indicate the convergence of the HPM to the exact solution. Namely, the invariant measures are analytically determined in several examples of chaotic mappings, what confirm the suitability and the applicability of the HPM in solving this kind of problem.

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